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The dynamics of antilock brake systems

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Abstract
The nonlinear dynamics of automobile braking are investigated. Nonlinearity arises because of the manner in which the friction coefficient between vehicle tyres and road surface depends upon vehicle speed and wheel angular speed. We show how antilock brake systems approach optimum braking performance.

1. Introduction

The frictional force F between a vehicle tyre and the road surface is a complicated function of vehicle speed u and tyre angular speed ω. In addition to these variables there are a number of parameters that influence F significantly, such as tyre air pressure, tread design and wear, and road surface. It has been determined empirically that contact friction of the form $F = \mu(\omega, u)N$ encapsulates experimental measurements obtained under laboratory conditions with constant u, ω. Here N is the normal force, which changes dynamically for each tyre if the vehicle acceleration is nonzero, for example if the vehicle is braking or cornering.

The tyre rim speed $w = \omega R$ (where R is effective tyre radius) equals vehicle speed u only if acceleration is zero. In this circumstance the normal force acting on a given tyre is constant in time, and the size of the *contact patch* of tyre on the road surface is also constant. No elements of the contact patch are slipping against the road surface. If the vehicle brakes, however, then some elements of the contact patch slip over the surface. The *longitudinal slip ratio* (henceforth simply *slip*)

$$s \equiv \frac{u - w}{u}, \qquad 0 \leqslant s \leqslant 1 \tag{1}$$

thus increases from $s = 0$ (no slipping). For a large applied brake torque, the tyres quickly tend to *lock up*, i.e. the wheels cease rotating ($s = 1$) and all elements of the contact patch are slipping over the road surface.

The friction coefficient μ has been found to depend upon the slip s only, and not upon w, u separately. (This is not quite true, as we shall see, but is a reasonable approximation for the purposes of analysis.) The form of the curve $\mu(s)$ has been parametrized in many

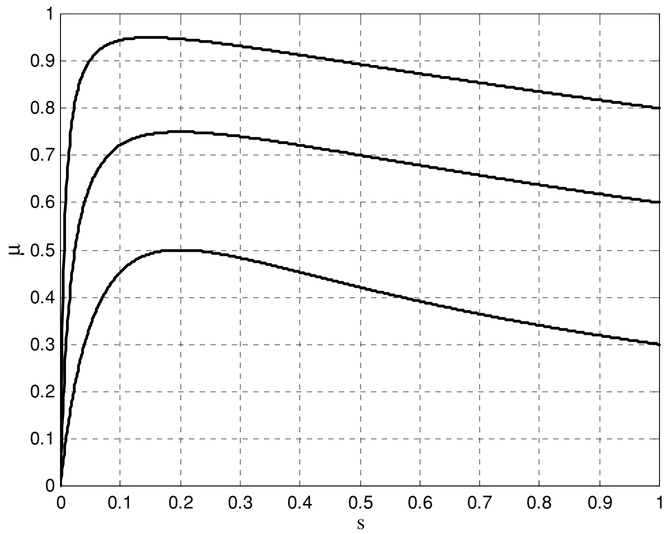

Figure 1. Friction coefficient versus slip for $(s_p, \mu_p, \mu_1) = (0.15, 0.95, 0.80)$, $(0.20, 0.75, 0.60)$ and $(0.20, 0.50, 0.30)$, as determined by equation (2). This form for $\mu(s)$ is applicable for a wide variety, but not all, of road conditions. Thus the top curve is typical for dry asphalt, whereas the bottom curve is more appropriate for wet asphalt roads.

different ways. The most widely quoted is the so-called *magic formula*, due to Pacejka [1], which applies to all cases of vehicle acceleration: braking, forward acceleration, yawing and cornering. In this paper we shall restrict our analysis to vehicle braking. With this restriction, we can ignore the dynamically changing contact patch, and instead consider a *lumped* dynamical friction model[1], in which we approximate the interaction between tyre and road as point contact friction with a friction coefficient that depends upon slip only.

We shall adopt a modified version of the $\mu(s)$ parametrization introduced by Kiencke and Daiss [2]:

$$\mu(s) = \frac{as}{b + cs + s^2}. \tag{2}$$

This is plotted in figure 1 for various parameter values; it summarizes experimental measurements very well. Note the manner in which friction coefficient changes with s. As the vehicle brakes, $\mu(s)$ increases from zero to reach a peak value $\mu_p = \mu(s_p)$ at $s_p =$ 0.15–0.20, and then falls away gradually to a value $\mu_1 = \mu(1)$ at lockup. The parameters a, b and c of equation (2) can be expressed in terms of μ_p, μ_1 and s_p:

$$a = \frac{\mu_p \mu_1}{\mu_p - \mu_1}(1 - s_p)^2 \qquad b = s_p^2 \qquad c = \frac{\mu_1\left(1 + s_p^2\right) - 2\mu_p s_p}{\mu_p - \mu_1}. \tag{3}$$

We shall find the parametrization of equation (2) to be pedagogically more useful than other parametrizations, such as the magic formula, since equation (2) permits certain results to be obtained analytically, rather than by numerical calculation.

From equations (1) and (2) it is clear that the dynamics of vehicle braking is nonlinear. It is also clear from the foregoing discussion and figure 1 that simply locking the wheels (non-ABS braking) is not optimum, since $\mu_1 < \mu_p$. In section 2 we present the equations of

[1] Lumped models lead to simple ordinary differential equations in time, whereas the much more complicated *distributed* models (contact patch) result in coupled partial differential equations in time and space [2].

motion for non-ABS braking, and show under what circumstances the wheels tend to lock up. Attempts to avoid locking lead to hysteresis, as we shall see.

In section 3 we investigate ABS, which is a feedback system providing rapid (usually electronically controlled) changes in applied braking torque to prevent wheel lockup. Thus from the form of $\mu(s)$ we see that a perfect ABS system would improve braking performance by maintaining the friction coefficient at its peak value μ_p. Non-ABS brakes almost always lock up quickly, so that for most of the braking period the friction coefficient is μ_1. A simple analysis shows that the stopping distance of a vehicle with perfect ABS is less than that of a vehicle without ABS by a factor μ_1/μ_p.

Practical ABS has been shown to reduce the stopping distance somewhat on dry asphalt surfaces, for which μ_1 is only slightly less than μ_p (see figure 1). It is much more successful on wet asphalt, but actually performs worse than non-ABS brakes on deformable surfaces such as gravel or packed snow[2] [3].

ABS and non-ABS vehicle braking provides an interesting and readily appreciated application of mechanics in general and of dynamical systems in particular at the introductory and intermediate university level. The simple mathematical models presented are readily accessible, and may be used as the basis of more detailed study projects.

2. Non-ABS dynamics

We make the following simplifying assumptions to reduce the number of parameters and to emphasize the underlying dynamics of braking:

1. The vehicle is moving along a straight line, and does not deviate from it.
2. We restrict attention to one wheel (the so-called *quarter-car* model).

Thus motion is restricted to one dimension, and we can ignore vehicle yaw and dynamical load transfer.

Given these simplifying assumptions, the equations of motion for a braking wheel are easily derived:

$$m\dot{u} = -\mu(s)N \qquad I\dot{\omega} = \mu(s)NR - G. \tag{4}$$

Here m is the (quarter) vehicle mass, $\mu(s)$ is the slip-dependent friction coefficient of equation (2), N is the normal force ($= mg$ here, given our assumptions, where g is the acceleration due to gravity), I is the wheel moment of inertia about the axle, and G is the applied brake torque.

It is straightforward to solve equations (4) numerically, but we shall instead follow Olson *et al* [4] who choose $(u,\ s)$ as the dynamical variables rather than $(u,\ \omega)$. This approach leads to greater insight. Much of the analysis of this section is taken from [4]; we apply it with our μ parametrization and so are able to obtain some of the results analytically, rather than numerically as in [4]. This enhances the pedagogical value of the analysis by making clear how our results depend upon friction parameters. From equations (1) and (4) it is not difficult to show that the equations of motion in terms of u and s are

$$\dot{u} = -\mu(s)g \qquad \dot{s} = \frac{g}{u}h(s) \tag{5}$$

where

$$h(s) \equiv \Gamma - (\nu + 1 - s)\mu(s). \tag{6}$$

[2] The dependence of friction coefficient upon slip is very different from that of figure 1 for deformable surfaces. It is thought that the poor ABS performance is because wheels that are sliding cause a 'dam' of gravel/ wet snow to build up in front of the wheels, thus aiding braking. ABS is designed to prevent sliding.

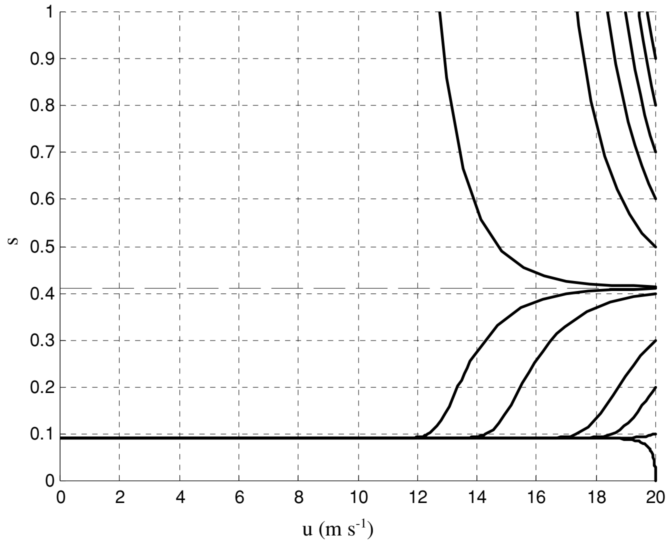

Figure 2. State space diagram for non-ABS braking with $s_p = 0.2$, $\mu_p = 0.5$, $\mu_1 = 0.3$, $\nu = 15$, $\Gamma = 7$, $u_0 = 20$ m s^{-1}. In this case the stable and unstable equilibrium points are $s_- = 0.09$, $s_+ = 0.41$ as determined by equation (8). The domain of attraction is $0 < s_0 < s_+$ where s_0 is the initial slip value. For $s_+ < s_0 < 1$ the wheel tends to lockup ($s = 1$).

Here

$$\Gamma \equiv \frac{GR}{Ig}, \qquad \nu \equiv \frac{mR^2}{I} \tag{7}$$

are dimensionless brake torque and moment of inertia. Since the wheel mass is much less than that of the vehicle we expect $\Gamma, \nu \gg 1$.

Note from (5) that the slip s is constant if $h(s) = 0$. If we denote by $s_\pm$ the equilibrium values of slip, $h(s_\pm) = 0$, then from our choice (2) of μ parametrization we can solve for $s_\pm$ analytically:

$$s_\pm = \frac{1}{2(\Gamma + a)}\left[a(\nu+1) - c\Gamma \pm \sqrt{[a(\nu+1) - c\Gamma]^2 - 4b\Gamma(\Gamma + a)}\right]. \tag{8}$$

We see from numerical integration of (5) that $s_+(s_-)$ represents a point of (un)stable equilibrium[3]. In the language of dynamical systems $s_+(s_-)$ is an attractor (a repellor). This is illustrated in the state space diagram of figure 2, where the domain of attraction is clear. The trajectories in state space proceed from high to low speeds and so, from equations (5), it is clear that the wheel must come to rest under steady-slip conditions (either $s = s_-$ or $s = 1$) since $\dot{s}$ changes very rapidly as $u \to 0$.

From equation (8) we see that there is a critical value s_{cr} for which $s_+ = s_-$:

$$s_{cr} = \sqrt{\frac{\Gamma_{cr}}{\Gamma_{cr} + a}}\, s_p \tag{9}$$

where Γ_{cr} is found by setting to zero the discriminant in (8)

$$\begin{aligned} \Gamma_{cr} &= \frac{a}{c^2 - 4b}\left[c(\nu+1) + 2b - 2\sqrt{b(\nu+1)^2 + bc(\nu+1) + b^2}\right] \\ &\approx \mu_p\mu_1\frac{(1 - 2s_p)}{\mu_1 - 2s_p\mu_p}(\nu+1), \qquad s_p \ll 1. \end{aligned} \tag{10}$$

[3] It is straightforward to verify this analytically by considering the influence on $\dot{s}$ of small changes δ in s, so $s \to s_\pm + \delta$. In [4] this result is obtained generally, for all forms of $\mu(s)$.

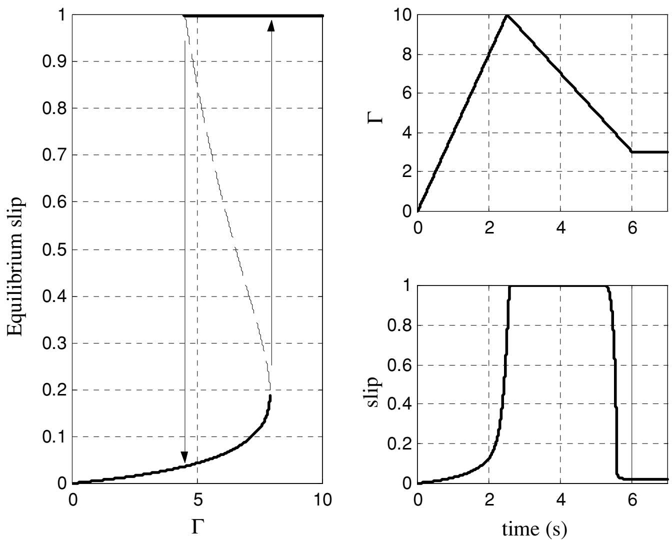

Figure 3. Hysteresis in non-ABS braking. Left: for the parameter values of figure 2, equilibrium slip is plotted as a function of brake torque. Stable equilibrium values (s_-) are shown as bold lines, unstable equilibrium (s_+) as the dashed line. Jumps are indicated by arrows. Top right: brake torque is varied slowly in time. Bottom right: when the torque exceeds 8, the slip jumps to $s_- = 1$ (lockup); as torque is lowered, the slip jumps to $s_- = 0.03$, but not until torque has been reduced to about $4\frac{1}{2}$.

(There is a second solution, which can be ignored since it leads to $s_\pm < 0$, corresponding to positive acceleration rather than braking.) If the braking torque exceeds this critical value, $\Gamma > \Gamma_{cr}$, then no constant-slip ($s = s_-$) braking can occur and the wheels lock up ($s = 1$). This increases the vehicle stopping distance because $\mu(s_-) > \mu_1$. For example, for the parameters of figure 2 we find $s_- = 0.09$ and so, from figure 1 or equation (1), $\mu(s_-) \approx 0.45$ whereas $\mu_1 = 0.30$.

Thus in the non-ABS case, the effectiveness of the brakes increases with applied torque up to $\Gamma = \Gamma_{cr}$; for larger values the wheels lock and braking deteriorates. If the wheels lock, a driver may lower brake torque (reduce Γ) so as to reduce slip from $s = 1$. However, the optimum equilibrium value s_p ($\approx s_{cr}$) will not apply, no matter how gently the driver eases off the brakes, because of hysteresis. This is illustrated in figure 3 where $s_-(\Gamma)$ is plotted (obtained by inverting equation (8)). This shows that, once Γ_{cr} is exceeded, the equilibrium slip jumps from s_{cr} to 1; subsequently reducing Γ leads to a new equilibrium slip. Γ must then be increased again to increase μ.

As noted in [4] and as seen explicitly in (9) we find $s_{cr} < s_p$ so that optimum braking performance (corresponding to μ_p) cannot be reached by gradually increasing Γ. In practice, however, this is unimportant since, for realistic parameter values, the difference between s_{cr} and s_p is small. For the parameters of figures 2 and 3 we find $s_{cr} \approx 0.19$ whereas $s_p = 0.20$.

Equation (10) shows us how critical braking torque depends upon the tyre/road friction characteristics. Thus Γ_{cr} will increase as peak friction coefficient increases and as vehicle mass increases, as we might expect. Less obviously, we see that Γ_{cr} will increase as s_p increases and as μ_1 decreases.

The fact of hysteresis (figure 3) shows that an adaptive antilock brake system, to which we now turn, must respond quickly once the wheels begin to lock, otherwise it will be unable to impose an effective friction coefficient close to optimum.

3. ABS dynamics

Any ABS system must include a sensor to show when a wheel is slipping over the road surface, so that brake torque can be reduced at the right time. The sensor will be required again to tell when the torque should once more be increased. If the sensor is accurate, and the braking system responds quickly enough, then we can anticipate that the friction coefficient will oscillate about the peak of figure 1 and vehicle braking will be close to optimum. In practical systems this cycling of reduced and then increased torque occurs up to 15 times per second, and gives rise to the term *cadence braking* to describe ABS.

Ideally the sensor should estimate wheel rotation rate ω and vehicle speed u, and so determine the slip. Measuring ω is easy and economical (usually achieved via EM induction) but measuring u is relatively expensive, and so practical ABS systems must be able to work only with wheel rotation rate. A rapid increase in slip rate, as the wheel locks during braking, corresponds to a rapid reduction in ω. Here we look at a simple algorithm that implements ABS by measuring wheel angular acceleration.

From equations (4)–(6) we see that

$$\ddot{\omega} = \frac{\nu g}{u}\mu'[\Gamma - (\nu + 1 - s)\mu], \qquad \mu' \equiv \frac{\mathrm{d}\mu}{\mathrm{d}s}. \tag{11}$$

We choose two brake torque values Γ_{min} and Γ_{max} to be applied by this ABS system. Clearly these should be chosen in such a way that

$$\Gamma_{\mathrm{min}} < (\nu + 1 - s)\mu < \Gamma_{\mathrm{max}}. \tag{12}$$

The algorithm we adopt is as follows:

$$\text{If } \ddot{\omega} < 0 \text{ then } \Gamma_{\mathrm{min}} \leftrightarrow \Gamma_{\mathrm{max}} \qquad \text{(simple ABS algorithm)}. \tag{13}$$

A moment's reflection shows how this will work. If $\Gamma = \Gamma_{\mathrm{max}}$ and $\ddot{\omega}$ is measured to be negative, then from (11) it must be that $\mu' < 0$ and so, from figure 1, the wheels are heading towards lockup, i.e. slip exceeds the optimum value and is (rapidly) increasing. In this situation we want to reduce brake torque to reduce slip, and so we set $\Gamma = \Gamma_{\mathrm{min}}$. Similarly if $\Gamma = \Gamma_{\mathrm{min}}$ and $\ddot{\omega} < 0$ then $\mu' > 0$ and so $s < s_{\mathrm{p}}$, from figure 1. Here we want to increase torque and so set $\Gamma = \Gamma_{\mathrm{max}}$. If $\ddot{\omega} > 0$ then we do not need to change Γ.

Solving equations (4) numerically and applying the simple ABS algorithm of (13), we find that braking performance is better than that obtained without ABS, assuming the same parameters. This is shown in figure 4. Note that wheel rim speed rises and falls as brake torque is changed. The resulting slip oscillates about the optimum value of $s_{\mathrm{p}} = 0.2$ (in this case), instead of rapidly heading to lockup ($s = 1$). Thus the friction coefficient exceeds μ_1 for most of the trajectory, and the stopping distance is reduced to about three-quarters of that obtained without ABS.

One shortcoming of the simple algorithm is that we have fixed values of Γ_{min} and Γ_{max}. Were this algorithm to be applied in practice, we would have to choose Γ_{min} small enough and Γ_{max} large enough to satisfy (12) for all possible braking environments. In many situations this will make the ABS response rather coarse and clumsy—note from figure 4 the large swings in wheel speed. It would be better if we could estimate friction parameters from measured data, so that the chosen values of applied brake torque reflect the braking environment. One straightforward method of achieving this is as follows. From the measured wheel acceleration, $\dot{w}$ is the friction coefficient and brake torques are estimated as follows:

$$\hat{\mu} = \frac{\Gamma + \dot{w}/g}{\nu}, \qquad \hat{\Gamma} = (\nu + 1 - \bar{s}_{\mathrm{p}})\hat{\mu}, \qquad \Gamma_{\substack{\mathrm{max}\\ \mathrm{min}}} = \hat{\Gamma} \pm 1. \tag{14}$$

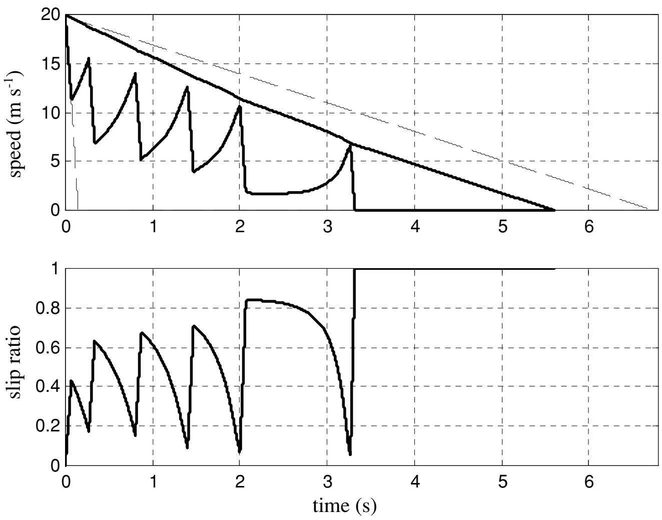

Figure 4. Non-ABS versus ABS braking. Here the initial speed is 20 m s^{-1} and friction characteristics are $s_p = 0.2$, $\mu_p = 0.5$, $\mu_1 = 0.3$, as in figure 1. The dimensionless moment of inertia is $\nu = 15$. Top: for non-ABS braking the dashed lines represent vehicle speed $u(t)$, ceasing after 6.8 s, and wheel rim speed $w(t)$, which falls to zero after 0.15 s. The dimensionless braking torque is $\Gamma = 20$. With these parameters the stopping distance is 67 m. For the simple ABS algorithm of equation (13), $u(t)$ and $w(t)$ are shown as bold lines. The minimum and maximum torques are fixed at $\Gamma_{\min} = 5$ and $\Gamma_{\max} = 20$. Now the wheel speed oscillates and takes 3.3 s to lock. Stopping distance is reduced to 51 m. Bottom: slip $s(t)$ for the ABS case.

The observed peak of μ(s) lies between 0.15 and 0.20 for most braking environments, and so we choose $\bar{s}_p = 0.17$. We have arbitrarily chosen $\Gamma_{\max} - \Gamma_{\min} = 2$ here. This value may need to be increased if we are not very confident about our estimate $\hat{\Gamma}$ of optimum torque (see equations (5) and (6)), or may be reduced if we associate a high confidence level with our estimate. Amending algorithm (13) with the dynamical estimates of required brake torque, given by (14), we further improve braking performance. This is illustrated in figure 5, assuming the same parameter values of figure 4.

The ABS algorithm of equations (13) and (14) significantly reduces the stopping distance for most combinations of parameter values that apply in the real world. One exception arises, unsurprisingly, when the friction coefficient $\mu(s)$ is insensitive to s for large s. We see such an example in figure 1, corresponding to a dry asphalt road surface. Here $\mu_p = 0.9$ and $\mu_1 = 0.8$; the difference is small and so there is not much benefit to be accrued by ABS. Our numerical calculations show that, in this case, the stopping distance is 25 m without ABS and improves only slightly to 23 m with ABS. Thus our simple model reproduces the observation that ABS is more effective on wet asphalt roads ($\mu_p = 0.5$, $\mu_1 = 0.3$) than on dry asphalt roads.

Our model is unrealistic for two reasons, but this does not alter the validity of our analysis or results, as we now show. Firstly, in the real world the friction coefficient is not a function of slip only. A more realistic parametrization is [2]

$$\mu(s, u) = \mu(s)\exp(-(u - u_0)/d) \tag{15}$$

with $\mu(s)$ given by equation (2), and with $d \sim 80$ m s^{-1} for a reference speed of $u_0 =$ 20 m s^{-1}. Incorporating this speed dependence in our calculations makes analysis more difficult, but does not significantly alter the results of calculations, as we see in figure 6. This is because the observed speed dependence parametrized in (15) simply scales μ(s), and does

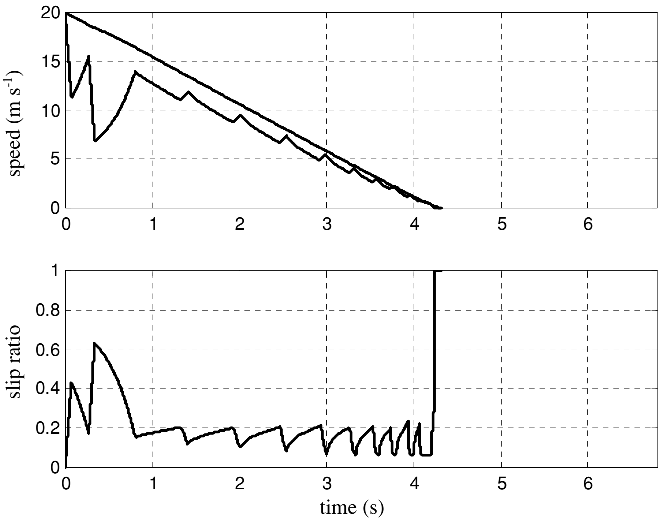

Figure 5. ABS braking for the parameters of figure 4, but here the applied torques $\Gamma_{\min,\max}$ are not fixed values, but rather are estimated dynamically by equation (14). The estimate is updated 15 times per second. Note how, compared to figure 4, slip ratio is less variable. The mean friction coefficient for this trajectory is 0.47, which is close to the peak value of $\mu_p = 0.50$. Stopping distance is reduced to 43 m.

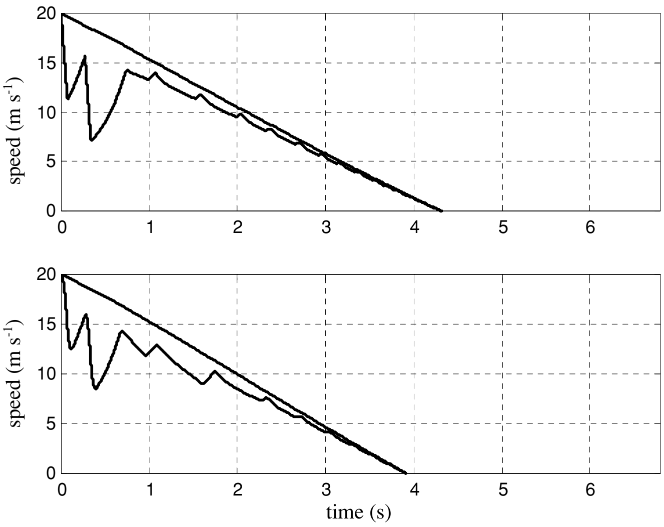

Figure 6. ABS braking for the parameters of figures 4 and 5, but here we include top the observed speed dependence of the friction coefficient (equation (15) with $u_0 = 20$ m s^{-1} and $d = 80$ m s^{-1}) and bottom smoothly varying torque (equation (16) with $\Delta t = 1/15$ s). The influences of these practical considerations upon our simple ABS model are small, as we see by comparison with figure 5.

not alter the shape of the curve. Secondly, in calculating the results plotted in figures 4 and 5 we have assumed that brake torque can be changed instantaneously. In practice these changes are continuous. If we model this by, for example,

$$\Gamma(t) = \frac{1}{2}(\Gamma_{\max} + \Gamma_{\min}) + \frac{1}{2}(\Gamma_{\max} - \Gamma_{\min}) \sin\left(\pi \frac{t - t_0}{\Delta t}\right), \qquad t_0 - \frac{1}{2}\Delta t < t < t_0 + \frac{1}{2}\Delta t \tag{16}$$

then the applied torque changes between $\Gamma_{\min}$ and $\Gamma_{\max}$ smoothly, over an interval Δt. We show the influence of this in figure 6. Comparing with figure 5 we see that there is only a slight difference.

4. Summary and discussion

The nonlinear dynamics of vehicle braking is a consequence of the slip dependence of the frictional force between the tyre and the road, as shown in figure 1. For non-ABS braking the wheels tend rapidly to lockup which, for the most common braking environments, is not optimum. There may be another steady-slip state (figure 2), but this also is sub-optimum, because the steady-slip value is usually not close to the peak value s_p. Attempting to modify braking torque dynamically leads to hysteresis, as in figure 3. We have shown how it is possible to estimate critical braking torque in terms of friction characteristic parameters (equation (10)).

We have outlined a simple ABS braking algorithm, and shown how this improves braking performance (figures 4 and 5). For simplicity this algorithm ignored the dependence of friction upon speed and the finite reaction time of practical braking systems. We have demonstrated (figure 6) that the influence of these practicalities upon the predictions of our ABS model is small.

Our analysis was based upon a number of idealizations: quarter-car model, lumped friction model and static friction characteristics. This last idealization consists of assuming that the curves of figure 1 apply during the braking process, even though applied torque may be changing rapidly during this time. This approach is generally adopted, though recently there have been a number of dynamical models developed [2], in which the friction force builds up to the static form of figure 1, as braking progresses.

We finish by noting some of the interesting statistical consequences of ABS braking for road traffic accidents. We have illustrated how ABS reduces stopping distances; it also improves vehicle steerability during braking. Despite these advantages, several US studies conducted in the 1990s show ‘close to zero’ net benefit [5]; there is no reduction in on-road crashes, and no reduction in the frequency or cost of crashes for vehicles equipped with ABS. This is unexpected, and may be due in part to the fact that not all vehicles have ABS as yet (e.g. an ABS car is less likely to rear-end the car in front, but more likely to get rear-ended). It may be due in part to driver behaviour. Thus, some drivers are less cautious while driving ABS cars. Some drivers persist in ‘pumping’ the brakes; this may be appropriate for non-ABS brakes (attempting to re-acquire optimum brake performance as in figure 3) but confuses an ABS system and reduces its effectiveness. Despite the lack of improvement in road accident statistics, however, ABS systems are installed in an increasing percentage of vehicles (currently 65% of new cars and 92% of light trucks in N America).

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